III. Probabilities in the Game of "Shooting Craps."

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Webster's *International Dictionary* gives the following explanation: "The caster throws or 'shoots' the dice, and wins if the throw is 7 or 11 (called a *nick* or *natural*), but loses if it is 2, 3, or 12 (called a *crap*). If 4, 5, 6, 8, 9, or 10 is thrown it becomes the caster's 'point,' and the caster continues to throw until he wins, by again throwing his point, or loses, by throwing 7. The odds are 251 to 244 against the caster."

This probability and the incidence of various points, craps, and naturals have been tested in a series of 9,900 games. Theoretical and actual incidences follow:

		Theoretical.		Actual.	
		Won.	Lost.	Won.	Lost.
Naturals	7	1,650		1,635	
	11	550		553	
Craps	2		275		259
	3		550		508
	12		275		293
Points	4	275	550	267	565
	5	440	660	481	662
	6	625	750	567	787
	8	625	750	620	738
	9	440	660	451	685
	10	275	550	297	532
Totals		4,880	5,020	4,871	5,029

In the further discussion of this game, use is made of the following theorem, which is believed to be new:

Theorem: *In any series of games where the probability of winning is constantly p, the average number of games won, up to and including the first lost game, is the reciprocal of the probability of losing.*

For the average number, A, is equal to the limit of the sum of the various probabilities that the first $(n-1)$ games are won and the nth lost multiplied by n, where n increases without limit, that is:

$$A = 1\cdot q + 2qp + 3qp^2 + \cdots + nqp^{n-1} + \cdots \text{ (where } q \text{ is the probability of losing)}$$

$$= q\cdot \sum_0^\infty (n+1)p^n, \qquad 0 < p < 1,$$

$$= q\cdot \left[\sum_0^\infty p^n\right]^2 = q\cdot\left[\frac{1}{1-p}\right]^2 = \frac{1}{q}$$

We employ this theorem to determine the average number of rolls needed to decide a game. For example, if "6" is a point, the point is won or lost as "6" or "7" is thrown, and since of the 36 ways in which 2 dice may fall, 5 ways give "6," and 6 give "7," the probability of deciding the point on the next cast is 11/36. Hence 36/11 throws, on the average, are required to decide the point "6." Exactly the same reasoning holds if "8" is the point. For "5" or "9,"

A is 18/5; for "4" or "10," A is 4. The average of these, allowing for their relative incidence is $3\frac{31}{55}$; and the average number of rolls per game (including those games won or lost by naturals or craps) can now be shown to be $3\frac{62}{165}$ In general, we may say that in 557 rolls of the dice: 55 are naturals or craps, 110 make points, and 392 complete these points.

This result may be shown independently of the theorem quoted. Let x equal the average number of rolls which are needed to complete a point *inclusive* of the first roll. Then in N games, where N is very large:

$$\frac{N}{3}+\frac{2Nx}{3}=N', \qquad \text{where } N' \text{ is the total number of rolls,}$$

$$=6\cdot s, \qquad \text{where } s \text{ is the total number of "7's" rolled,}$$

$$=6\cdot\left(\frac{N}{6}+\frac{N}{9}+\frac{2N}{15}+\frac{5N}{33}\right)$$

whence x equals $4\frac{31}{55}$, and the conclusion follows as above.

According to best Army traditions, the caster retains the dice, and has the privilege of naming the size of the next bet as long as he wins, and also if he throws a crap; giving the dice and the privilege to his opponent only if he loses a point. The probability that he holds the dice is obviously 244/495 augmented by the probability of throwing a crap (4 chances in 36), that is, 299/495. The average number of games which he will roll is then 495/196 or 2.5255, of which he will win 1.2449 games, and lose 1.2806 games.

De Morgan[1] indicated the incidence of unusual runs of luck in several series of 2048 trials where the probability p was 1/2. In this question of holding the dice we have a probability of 299/495, practically 3/5. The first 2,471 games in the test mentioned above comprise exactly 1,000 turns, and an integral approximation of the theoretical together with the actual incidence is tabulated below:

	Theoretical.	Actual.
Total number of games	2,526	2,471
Total number of turns	1,000	1,000
Turns ending with 1 game	396	423
" " " 2 games	239	210
" " " 3 "	145	142
" " " 4 "	87	93
" " " 5 "	53	70
" " " 6 "	32	23
" " " 7 "	19	14
" " " 8 "	12	9
" " " 9 "	7	3
" " " 10 "	4	6
" " " 11 "	2	1
" " " 12 "	2	4
" " " 13 "	1	2
" " " 14 " (or more)	1	0
	1,000	1,000

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[1] Augustus de Morgan, *A Budget of Paradoxes*, 2d ed., 1915, Vol. 1, pp. 281–3.